substituting Eq. (10) into Eq. (8), resulting in

$$R^* = [0.7(\sigma_\alpha/\sigma_l)/\alpha]^{1/2} \tag{14}$$

Note that the value of R^* is independent of K_t. For $\alpha = 1$, the condition when $\sigma_n = \sigma_l$, $R^* = \sqrt{0.7} = 0.8367$ for all values of the exponent m. For values of σ_n greater than σ_l, R^* is relatively insensitive to m as compared with its sensitivity to α.

Design Graphs

A sample design graph for a material whose Ramberg-Osgood exponent, $m = 10$, is shown in Fig. 1. Other similar graphs could be constructed for a range of m values. The graph allows a rapid determination of one of the three variables (σ_0/σ_l), K_t, or R, when two are specified. As an adjunct to the data in Fig. 1, the variation of σ_0/σ_l and K_t for constant values of α is shown in Fig. 2. Each curve on Fig. 2 represents the condition for a constant value of the discontinuity stress equal to the indicated Ramberg-Osgood end-value. If the reference stress is elastic, the associated reduction factor will be a constant whose value is a function of the end-strain ratio α given by Eq. (14). For reference stress in excess of the effective proportional limit, the reduction factor increases. Values of the reduction factor for all elastic reference stresses and for the reference stress at the secant-yield stress are given in Fig. 2.

A computer program has been written for calculation of R, σ_0/σ_l, σ_n/σ_l, and K_σ for a series of elastic concentration-factor values from 1.05 to 10 and for values of the Ramberg-Osgood exponent from 5 to 200. The data for Figs. 1 and 2 were obtained from the tabular output of the program.

Conclusions

1) Design relations have been developed for plastic discontinuity stresses using Neuber's equation and the Ramberg-Osgood analytic approximation of stress-strain properties.

2) The limiting value of the discontinuity stress in the developed relations is the Ramberg-Osgood end-value. The latter is the maximum value of stress for which the Ramberg-Osgood equation provides a satisfactory fit of experimental stress-strain data. The end-point of the equation has been identified using a non-dimensional strain parameter. The end-stress is obtained by computation.

3) The relative reduction of the stress concentration factor associated with plastic discontinuity stresses is independent of the elastic concentration factor. The reduction is mainly dependent on the maximum discontinuity stress and the Ramberg-Osgood exponent value.

4) Limiting values of the stress-concentration reduction factor occur when the discontinuity stress is at the Ramberg-Osgood end-value and the reference stress is elastic: a) If the end-value is the secant-yield stress, the plastic concentration factor is 83.7% of the elastic concentration factor for all values of the Ramberg-Osgood exponent and for all elastic values of the reference stress; or b) If the end value is greater than the secant-yield stress, the reduction value is a function of both the exponent and the end-stress. However, for given values of the two parameters, the limiting value of the reduction factor is constant for all elastic values of the reference stress.

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Some Singular Aspects of Three-Dimensional Transonic Flow

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Introduction

THIS Note discusses certain singular aspects in the steady formulation for three-dimensional transonic flows. The method of inner and outer expansions is used to show how two separate limits can be distinguished for the inner crossflow problem, the first leading to Laplace's equation and the second leading to a mixed-type equation. The particular equation used in any problem is fixed by the relative value of an aspect ratio A to a measure ϵ of the flow nonlinearity. The required matching process then determines the role of the outer small disturbance equation (SDE) in calculating near-field surface pressures.

Analysis

The classical SDE for steady three-dimensional transonic flow is useful in describing the near-isentropic flow about only certain types of thin wings. The most severe constraint appears to be that for near two dimensionality, that is, $A\tau^{1/3} \gg 1$, where τ is the thickness ratio. For slender, highly swept wings with $A\tau^{1/3} \ll 1$, the SDE does not seem to produce very encouraging results. For example, the three-shock pattern observed experimentally on swept supercritical wings (consisting of a conical forward shock, a rear shock, and a tip shock) cannot be predicted. This situation has forced a number of authors to reassess the usefulness of the SDE and a number of heuristic corrections have been proposed,[1,2] these models typically adding terms that contain various quadratic and cubic nonlinearities, spanwise terms, and so on. The most critical step implicit in these schemes assumes that the new equation describes *both* "inner" near-field and "outer" far-field flows. This is certainly not clear a priori since the problem must be studied using the complete equation. The failure of classical theory for certain configurations suggests that A must be included in the transonic limiting process. In the classical derivation[3] all space coordinates are normalized by the same reference length; only after the SDE is obtained are the well-known similarity rules involving A obtained. Thus, the approach taken here follows the spirit in which constant density slender body and planar wing theories are derived from the three-dimensional Laplace equation. The full potential equation is considered with A included at the outset, and different near-field equations are derived corresponding to different limiting processes.

The exact dimensional equation for the disturbance potential $\varphi(x,y,z)$ contains quadratic and cubic nonlinearities that may not be negligible. Let M_∞ denote the subsonic freestream Mach number, U_∞ the freestream speed, γ the ratio of specific heats; the streamwise, spanwise, and normal coordinates here being x,y, and z. Barred nondimensional

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variables are introduced with the definitions $x=C\bar{x}$, $y=S\bar{y}$, $z=L\bar{z}$, and $\varphi=\varphi_0\bar{\varphi}$, where C is the maximum chordwise extent, S is the span, and L and φ_0 are normalization constants to be discussed. Two nondimensional parameters that appear are $A=S/C$ and $\epsilon=\varphi_0/U_\infty C$, but different approximations are possible for different L's. In the first case $A\gg 1$ and $L=C$ is the relevant length scale; this leads to an equation (hereafter denoted by I) in which the spanwise terms are small. However, the proper length scale for the small A limit is $L=S$. The resulting equation (II) here leads to Laplace's equation for the crossflow if all nonlinearities are ignored.

Consider first the large A limit; two expansions for the potential can be introduced, an inner one satisfying tangency conditions and an outer one satisfying regularity conditions. Now for transonic flows the disturbances extend far laterally, implying an outer expansion of the form (all bars will be dropped) $\varphi=\varphi_o(x,y,\zeta)$ where $\zeta=\delta z$ is $O(1)$ in the far-field and $\delta\ll 1$ is determined by matching; the derivatives $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial\zeta$ here are all $O(1)$. As usual assume that A^{-2} and $1-M_\infty^2$ are both $O(\epsilon)$, set $\delta^2=M_\infty^2\epsilon(\gamma+1)$, and expand $\varphi_0=\varphi_o^{(1)}+\epsilon\varphi_o^{(2)}$. Since $A\gg 1$, retention of leading terms in I gives the usual transonic SDE. For the inner region we choose $\varphi=\varphi_i(x,y,z)$ where the undistorted variables are $O(1)$ near the wing. Substitution of $\varphi_i=\varphi_i^{(1)}+\epsilon^{1/2}\varphi_i^{(2)}$ into I leads to $\partial^2\varphi_i^{(1)}/\partial z^2=\partial^2\varphi_i^{(2)}/\partial z^2=0$ with the solutions $\varphi_i^{(1)}=zg(x,y)+h(x,y)$ and $\varphi_i^{(2)}=za(x,y)+b(x,y)$. The matching here is almost trivial and only a brief discussion will be given. The limit matching principle

$$\epsilon^{3/2}M_\infty(\gamma+1)^{1/2}\frac{\partial\varphi_o^{(1)}}{\partial\zeta}\Bigg|_{\zeta=0}\equiv(\epsilon\varphi_{i_z}^{(1)}+\epsilon^{3/2}\varphi_{i_z}^{(2)})\Bigg|_{z=\infty}$$

applied to $\epsilon\varphi_z$ shows that $\partial\varphi_i^{(1)}/\partial z=0$, the vertical velocity thus being of $O(\epsilon^{3/2})$. It follows that $g=0$ and $\varphi_i^{(1)}(x,y,\infty)=h(x,y)=\varphi_o^{(1)}(x,y,0)$, demonstrating how surface pressures can be obtained by using the outer expansion $\varphi_o^{(1)}$. Boundary conditions for this outer problem are easy to determine. Matching horizontal velocities leads to $b=0$ since no term of $O(\epsilon^{3/2})$ appears in the outer expansion. Because spanwise effects are higher order we can approximate in the foregoing equation $\epsilon^{3/2}\partial\varphi_i^{(2)}/\partial z=\epsilon^{3/2}a(x,y)\equiv\tau\partial f/\partial x$, $z=\tau f(x,y/A)$, say, being the wing surface. A nontrivial solution for the outer flow is obtained by choosing $\epsilon^{3/2}M_\infty(\gamma+1)^{1/2}=\tau$, with the result that $\delta=(M_\infty^2\tau(\gamma+1))^{1/3}$. For large A all of the classical results for the three-dimensional SDE apply. The degeneracy of the inner problem for large A's here permits us to solve the SDE using tangency conditions, but this need not be the case for small A's.

For small aspect ratios the preferred normalization is $L=S$. The relevant outer variables are x, $\zeta=\delta z$ and $\xi=\delta y$, with $\varphi=\varphi_o(x,\xi,\zeta)$ where again derivatives with respect to these variables are $O(1)$. We substitute the expansion $\varphi_o\cong\varphi_o^{(1)}+\epsilon\varphi_o^{(2)}$ into II, under the assumption that $1-M_\infty^2$ and δ^2/A^2 are both $O(\epsilon)$. Retention of leading order terms gives the classical equation

$$(1-M_\infty^2)\varphi_{o_{xx}}^{(1)}+\frac{\delta^2}{A^2}(\varphi_{o_{\xi\xi}}^{(1)}+\varphi_{o_{\zeta\zeta}}^{(1)})=\epsilon M_\infty^2(\gamma+1)\varphi_{o_x}^{(1)}\varphi_{o_{xx}}^{(1)}\qquad(1)$$

The inner expansion is $\varphi_i\cong\varphi_i^{(1)}(\epsilon x,y,z)+\lambda(\epsilon)\varphi_i^{(2)}(\epsilon x,y,z)$ where y and z are undistorted, and an explicit weak dependence on x is shown. Now the appearance of A can be eliminated from II written for φ_i, since we have required $A^2=\delta^2/\epsilon$. With $1-M_\infty^2$ being $O(\epsilon)$ the crossflow term $\epsilon/\delta^2(\varphi_{i_{yy}}+\varphi_{i_{zz}})$ dominates $(1-M_\infty^2)\varphi_{i_{xx}}$ for all small values of δ, showing how the near-field flow is essentially two dimensional. However different choices can be made for the stretching $\delta=\delta(\epsilon)$, and it is clear on examination of the full perturbation equation that two possibilities exist, the first being $\delta=\epsilon$ and the second being $\delta=\epsilon^{3/2}$. The first scaling $\delta=\epsilon$ leads to Laplace's equation for both $\varphi_i^{(1)}$ and $\varphi_i^{(2)}$. The matching proceeds as before and reproduces the well-known classical results. The second choice $\delta=\epsilon^{3/2}$, however, takes into account local compressibility effects that can become important for increasing nonlinearities. Under the assumed scalings for φ_i, the governing equation correct to second order becomes

$$\left[1-M_\infty^2\left(\frac{\gamma+1}{2}\varphi_{i_y}^2+\frac{\gamma-1}{2}\varphi_{i_z}^2\right)\right]\varphi_{i_{yy}}-2M_\infty^2\varphi_{i_y}\varphi_{i_z}\varphi_{i_{zy}}$$
$$+\left[1-M_\infty^2\left(\frac{\gamma+1}{2}\varphi_{i_z}^2+\frac{\gamma-1}{2}\varphi_{i_y}^2\right)\right]\varphi_{i_{zz}}=0\qquad(2)$$

where all of the coefficients shown earlier are $O(1)$ in the inner flow. For $\gamma=7/5$ it is possible to show that Eq. (2) is elliptic whenever $(\varphi_{i_y}^2+\varphi_{i_z}^2)M_\infty^2$ is less than 5/6 or greater than 5. Within this range the crossflow equation is hyperbolic, hence suggesting the possibility of supersonic flow and shock formation. In such an event it is unlikely that the outer solution by itself can be used to calculate near-field surface pressures, as was possible in the degenerate large A case.

The foregoing mixed-flow character is interesting because it shows how the crossflow becomes compressible when $\epsilon\sim A$, for a given $A\ll 1$. As the nonlinearity ϵ decreases, in particular, $\epsilon\sim A^2$, the flow becomes essentially incompressible. Thus, Eq. (2) embodies the classical incompressible approximation but it is not restricted to very small nonlinearities. It is not certain, however, whether or not Eq. (2) in general can be matched to Eq. (1); it is possible that intermediate expansions may be needed. The new equation given here contains cubic nonlinearities neglected in the usual derivation, and it is important to recognize their importance for certain flow configurations. Classical theory neglects these terms at the outset and therefore implicitly assumes $\epsilon\sim O(A^2)$.

Discussion and Summary

The preceding discussion shows how different equations governing the inner flow can be obtained depending on the relative scaling between ϵ and A; the inclusion of cubic nonlinearities here allows the possibility of supersonic flow and shock waves in the crossflow plane. This event must be allowed for in the matching procedure to predict surface pressures, shock strength, and shock position correctly. The complications discussed here disappear, however, if the full potential equation is used directly, since it is uniformly valid.

Some simple results can be obtained for axisymmetric bodies since the crossflow equation depends only on the radial coordinate $r=\sqrt{z^2+y^2}$ (this degeneracy results in the loss of mixed-type solutions). The governing equation for the outer flow is just Eq. (1) with the second term replaced by

$$\frac{\delta^2}{A^2}\left(\varphi_{o_{\rho\rho}}^{(1)}+\frac{1}{\rho}\varphi_{o_\rho}^{(1)}\right),\quad\rho=\delta r$$

It is solved with boundary conditions that depend on the choice of the inner flow. Let us assume the perturbation potential $\epsilon\varphi_i$ in the form

$$\epsilon\varphi_i\cong\epsilon\varphi_i^{(1)}(\epsilon x,r)+\epsilon\delta\varphi_i^{(2)}(\epsilon x,r)$$

where a weak x dependence is indicated. The matching

$$\lim_{\rho\to 0}\epsilon\delta\frac{\partial\varphi_o^{(1)}(x,\rho)}{\partial\rho}=\lim_{r\to\infty}\epsilon\left[\frac{\partial\varphi_i^{(1)}}{\partial r}+\delta\frac{\partial\varphi_i^{(2)}}{\partial r}\right]$$

of radial velocities implies that $\partial\varphi_i^{(1)}/\partial r(\epsilon x,\infty)=0$ and $\varphi_{o_\rho}^{(1)}(x,0)=\varphi_{i_r}^{(2)}(\epsilon x,\infty)$. In the classical case $\varphi_i^{(1)}$ satisfies Laplace's equation; the solution $\varphi_i^{(1)}=A(x)\log r+B(x)$

fulfills the first condition and the second states that tangency conditions are applied to the $\varphi_i^{(2)}$ problem. Thus, $r\varphi_{i_r}^{(1)} = A(x)$ must vanish, since it vanishes at the body surface. It follows, on matching potentials, that $\varphi_i^{(1)} = B(x)$ equals $\varphi_o^{(1)}(x,0)$. On the body $r=\tau R(x)$ surface pressures can be evaluated using $\epsilon^2\varphi_{i_r}^2 = \tau^2 R'^2$ and the result that $\varphi_{i_x}^{(1)}(\epsilon x, r) = \varphi_{o_x}^{(1)}(x,0)$. To evaluate the right side of the foregoing equation we need the solution for $\varphi_i^{(2)}$. The choice $\delta=\epsilon^{1/2}$ or $\delta=\epsilon$ leads to Laplace's equation with the solution $\varphi_i^{(2)} = C(x)\log r + D(x)$ which satisfies tangency conditions. Evaluation on the body surface leads to $\lim_{r\to 0}\epsilon\delta r\varphi_{i_r}^{(2)} = \epsilon\delta C(x) = \tau^2 RR'$, and so,

$$\lim_{\rho\to 0} \epsilon\rho \frac{\partial\varphi_o^{(1)}(x,\rho)}{\partial\rho} = \lim_{r\to\infty} \epsilon\delta r \frac{\partial\varphi_i^{(2)}}{\partial r} = \tau^2 RR'$$

since $\epsilon\delta r\varphi_{i_r}^{(2)}$ is independent of r. Hence $\epsilon=\tau^2$ as opposed to $\epsilon\sim\tau^{2/3}$ in the near-planar two-dimensional case.

The stretching $\delta=\epsilon^{3/2}$ is the one of interest and leads to the following crossflow equation for $\varphi_i(\epsilon x, r)$,

$$\left(1-\frac{6}{5}M_\infty^2\varphi_{i_r}^2\right)\varphi_{i_{rr}} + \left(1-\frac{1}{5}M_\infty^2\varphi_{i_r}^2\right)\frac{\varphi_{i_r}}{r} = 0$$

with the solution

$$r\varphi_{i_r}\left|1-\frac{1}{5}M_\infty^2\varphi_{i_r}^2\right|^{5/2} = \mathcal{C}(x)$$

The correct branch is the one that identifies with classical theory for large r. For small r's the compressible correction removes the usual logarithmic singularity in φ_i. The effect of this nonlinearity, important near the body, must be transferred to the outer flow. This is accomplished by first determining $\mathcal{C}(x)$ by evaluating the preceding equation on the body surface, that is,

$$\mathcal{C}(x) \cong \frac{\tau^2}{\epsilon} RR' \left[1-\frac{M_\infty^2\tau^2R'^2}{5\epsilon^2}\right]^{5/2}$$

Next the same function is evaluated for the outer flow, giving

$$\rho\varphi_{o_\rho}^{(1)}\left[1-\frac{M_\infty^2\delta^2}{5}\varphi_{o_\rho}^{(1)2}\right] \cong \rho\varphi_{o_\rho}^{(1)} + O(A^3)$$

Choosing $\epsilon=\tau^2$ produces the boundary condition for the outer flow,

$$\lim_{\rho\to 0} \rho\varphi_{o_\rho}^{(1)}(x,\rho) \cong RR'\left(1-\frac{M_\infty^2R'^2}{5A}\right)^{5/2}$$

where $M_\infty^2R'^2/5A<1$. The outer flow therefore sees an effective body slope reduced from its classical value and which, for small enough A's, decreases to zero. As before it is possible to show that $\varphi_i^{(1)} = \varphi_o^{(1)}(x,0)$.

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Driver Gas Contamination in a High-Enthalpy Reflected Shock Tunnel

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It is well known that the utility of reflected shock tunnels is seriously limited by premature driver gas contamination of the test gas.[1] Davies and Wilson[2] have developed an explanation of this effect, which is consistent with measurements made in relatively low-enthalpy shock tunnels, where the primary shock Mach number was less than 6 (Refs. 2 and 3). However, their theory indicates that early contamination should not occur if the shock tunnel is operated at primary shock Mach numbers which are less than a value near the tailored interface level. In experiments with a high-enthalpy shock tunnel,[4] it has been found that early contamination persisted at shock Mach numbers down to 60% of the tailored interface value. In this Note, their theory is extended to take account of these results.

Davies and Wilson base their explanation of early contamination on the bifurcation which occurs at the foot of the reflected shock wave as it interacts with the wall boundary layer in the shock tube. As shown in Fig. 1a, if the bifurcation persists as the reflected shock is transmitted through the driver gas, then the gas which passes through the bifurcation region is decelerated through two oblique shocks, thereby suffering a smaller change in velocity than the gas which passes through the normal reflected shock. Thus, the gas which passes through the bifurcation region has a velocity towards the contact surface, which causes it to penetrate the contact surface as a jet along the walls, and thereby, to contaminate the test gas. Now, bifurcation persists if the minimum stagnation pressure P_{st} in the boundary layer is such that some of the boundary-layer fluid cannot negotiate the shock pressure rise. Davies and Wilson calculated P_{st} from the Rayleigh supersonic pitot formula, i.e.,

$$P_{st}/P_3 = [(\gamma+1)M_b^2/2]^{\gamma/(\gamma-1)}[2\gamma M_b^2/(\gamma+1) - (\gamma-1)/(\gamma+1)]^{1/(1-\gamma)}$$

where P_3 is the pressure ahead of the transmitted shock, γ is the ratio of specific heats of the boundary-layer gas, and M_b is the minimum Mach number in the boundary layer with respect to the transmitted shock. They assumed that the boundary layer was composed of test gas, and M_b was the value at the wall. As already noted, this yielded results which correlated satisfactorily with experiments at low enthalpies.

The high-enthalpy experiments[4] were performed with helium driver gas, at primary shock Mach numbers ranging from 13 to 23. Piezoelectric transducers were used to measure the speed of the transmitted shock system u_T after the contact surface had come to rest following completion of the shock reflection process. Estimates had indicated that bifurcation was more probable at this speed than at any earlier stage in formation of the transmitted shock. The pressure ratio across this transmitted shock system P_T/P_3 was obtained by combining measurements of the pressure after shock reflection with the contact surface pressure, as calculated

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